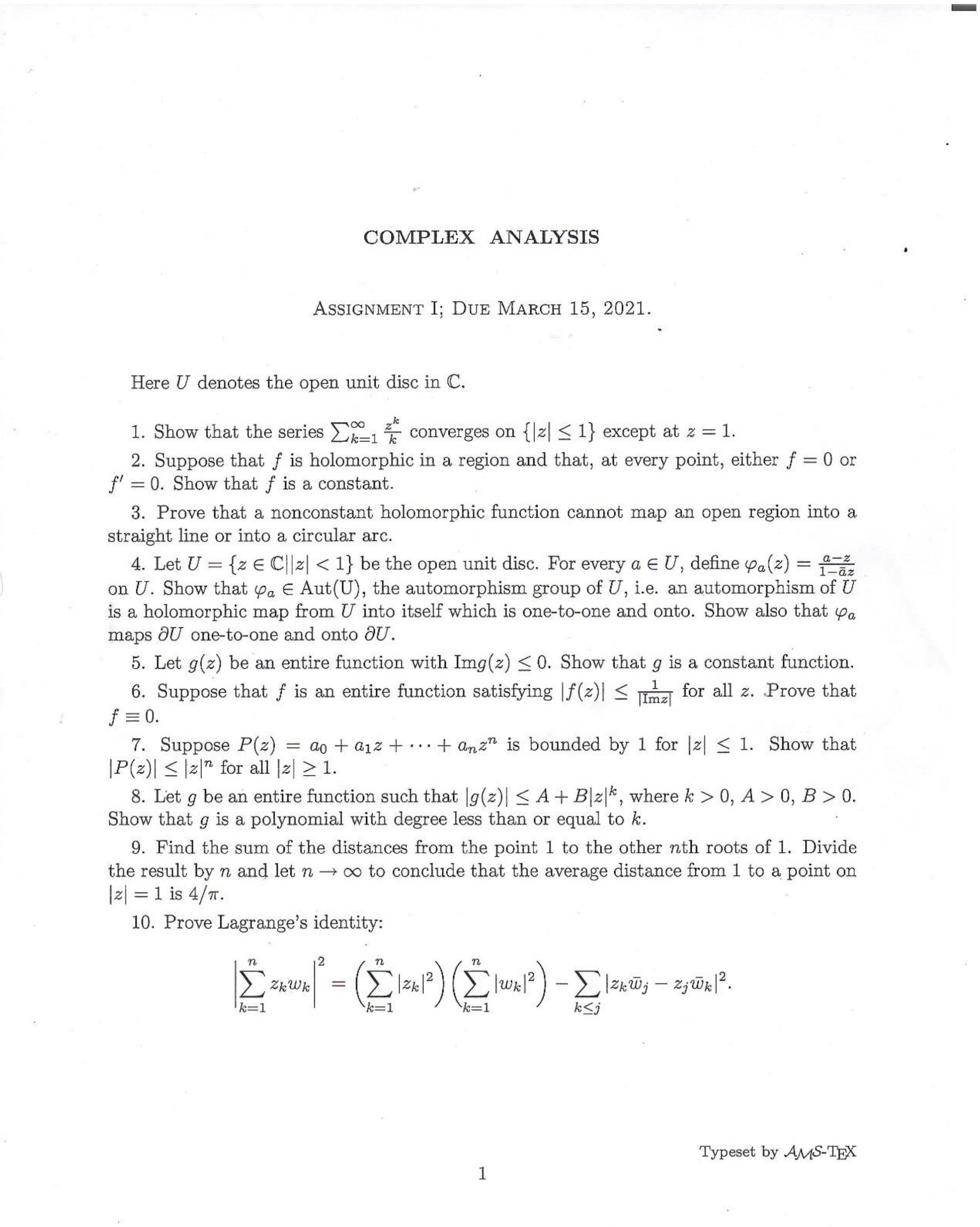
【10920 程守慶教授複變數函數論 / 第 3 堂版書】

【3/4 本周作業】



COMPLEX ANALYSIS

Assignment I; Due March 15, 2021.

Here U denotes the open unit disc in $\mathbb{C}$.

1. Show that the series $\sum_{k=1}^{\infty} \frac{z^k}{k}$ converges on $\{|z| \le 1\}$ except at $z = 1$.

2. Suppose that f is holomorphic in a region and that, at every point, either $f = 0$ or $f' = 0$. Show that f is a constant.

3. Prove that a nonconstant holomorphic function cannot map an open region into a straight line or into a circular arc.

4. Let $U = \{z \in \mathbb{C} | |z| < 1\}$ be the open unit disc. For every $a \in U$, define $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ on U. Show that $\varphi_a \in \mathrm{Aut(U)}$, the automorphism group of U, i.e. an automorphism of U is a holomorphic map from U into itself which is one-to-one and onto. Show also that φ_a maps ∂U one-to-one and onto ∂U.

5. Let $g(z)$ be an entire function with $\mathrm{Im} g(z) \le 0$. Show that g is a constant function.

6. Suppose that f is an entire function satisfying $|f(z)| \le \frac{1}{|\mathrm{Im} z|}$ for all z. Prove that $f \equiv 0$.

7. Suppose $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is bounded by 1 for $|z| \le 1$. Show that $|P(z)| \le |z|^n$ for all $|z| \ge 1$.

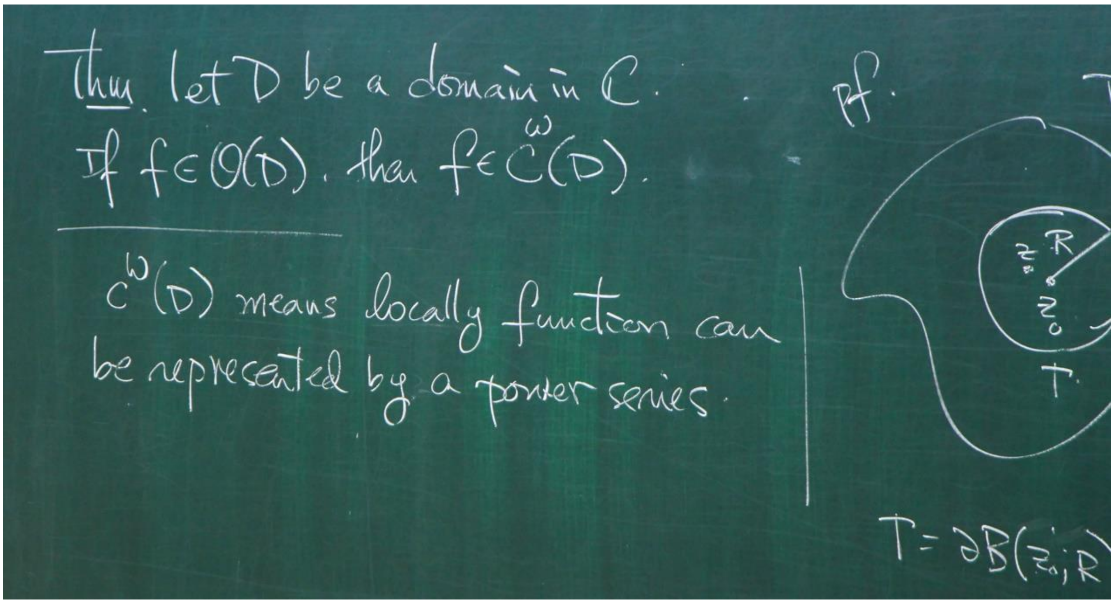
8. Let g be an entire function such that $|g(z)| \le A + B|z|^k$, where $k > 0$, $A > 0$, $B > 0$. Show that g is a polynomial with degree less than or equal to k.

9. Find the sum of the distances from the point 1 to the other nth roots of 1. Divide the result by n and let $n \to \infty$ to conclude that the average distance from 1 to a point on $|z| = 1$ is $4/\pi$.

10. Prove Lagrange's identity:

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 = \left(\sum_{k=1}^{n} |z_k|^2\right)\left(\sum_{k=1}^{n} |w_k|^2\right) - \sum_{k \le j} |z_k \bar{w}_j - z_j \bar{w}_k|^2.$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

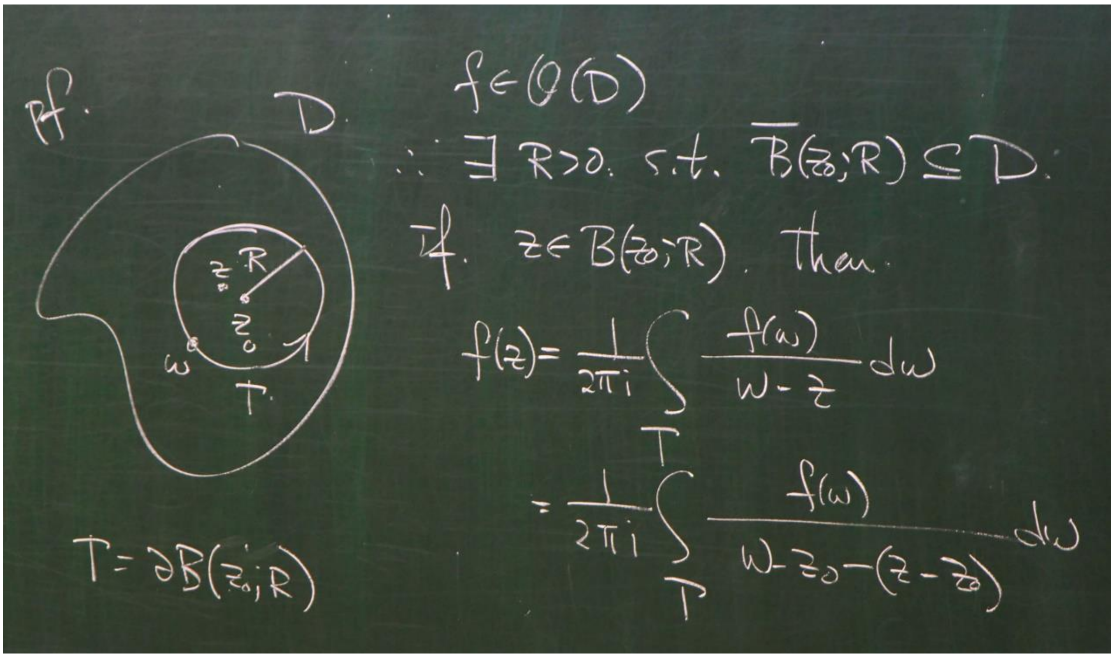


Thm. let D be a domain in $\mathbb{C}$.
If $f \in \mathcal{O}(D)$, then $f \in C^{\omega}(D)$.

$C^{\omega}(D)$ means locally function can be represented by a power series.

pf.

$T = \partial B(z_0; R$



pf.

$T = \partial B(z_0; R)$

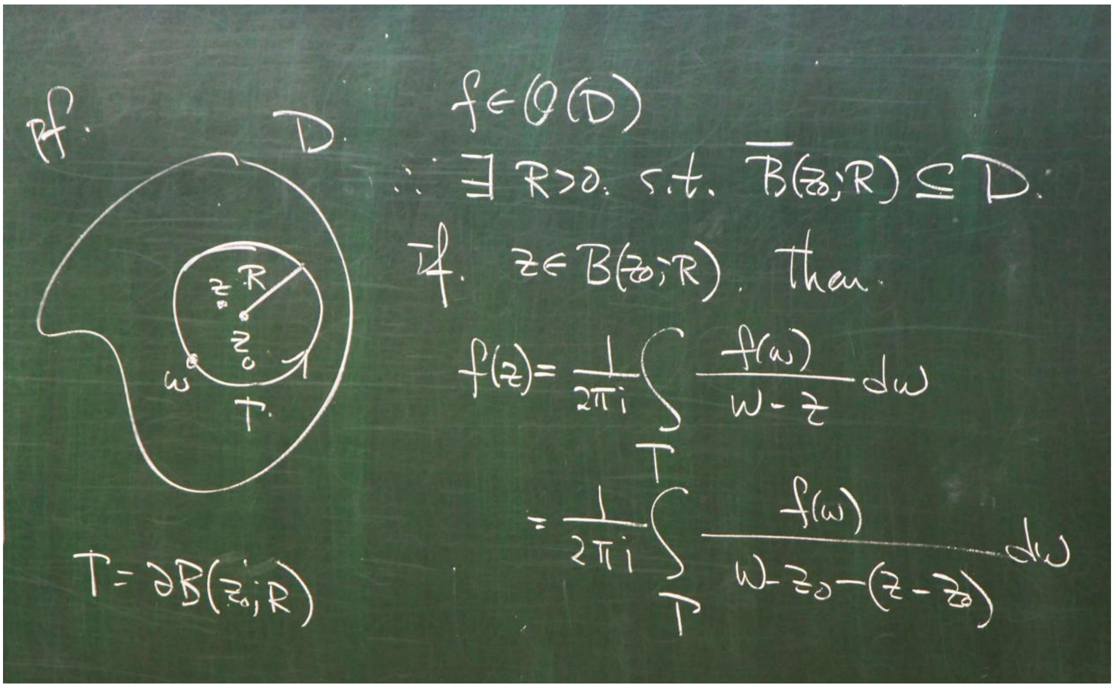
$f \in \mathcal{O}(D)$

$\therefore \exists\, R > 0$ s.t. $\overline{B}(z_0; R) \subseteq D$.

If $z \in B(z_0; R)$, then

$$f(z) = \frac{1}{2\pi i}\int_T \frac{f(w)}{w - z}\,dw$$

$$= \frac{1}{2\pi i}\int_T \frac{f(w)}{w - z_0 - (z - z_0)}\,dw$$



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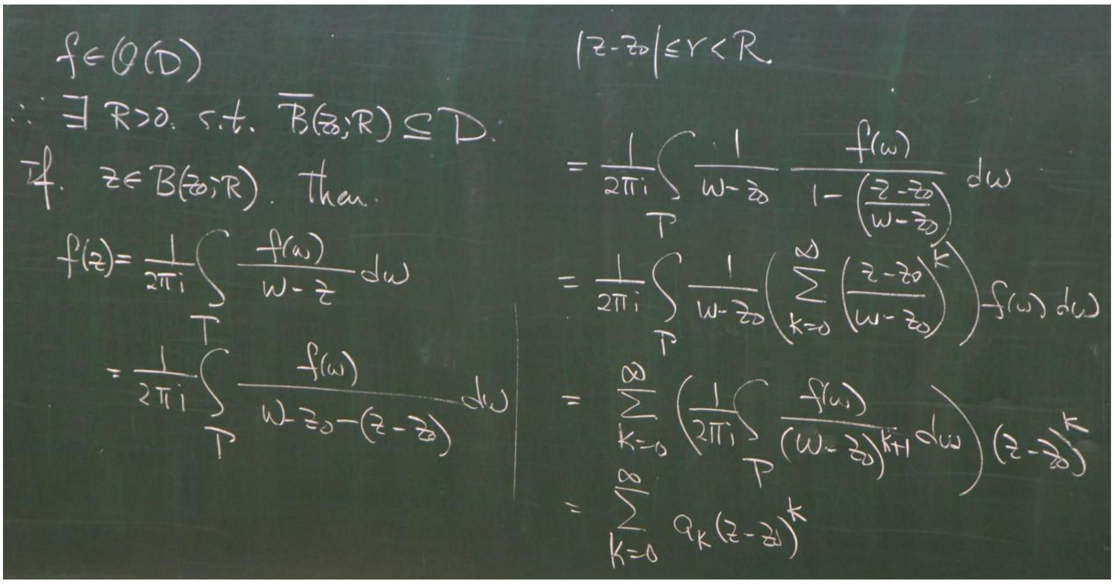

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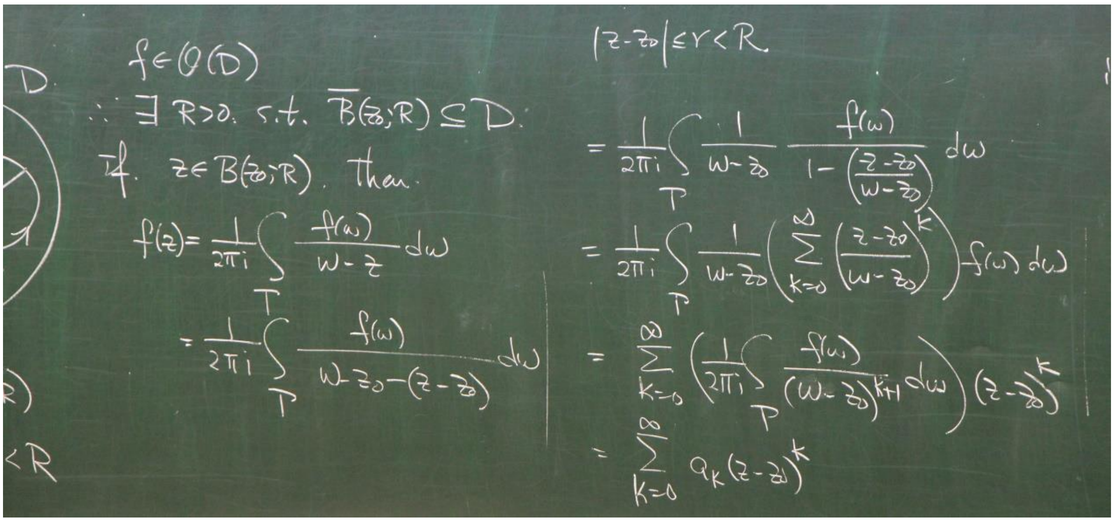
$|z-z_0| \le r < R.$

$$=\frac{1}{2\pi i}\int_{T}\frac{1}{w-z_0}\,\frac{f(w)}{1-\left(\frac{z-z_0}{w-z_0}\right)}\,dw$$

$$=\frac{1}{2\pi i}\int_{T}\frac{1}{w-z_0}\left(\sum_{k=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^{k}\right)f(w)\,dw$$

$$=\sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\int_{T}\frac{f(w)}{(w-z_0)^{k+1}}\,dw\right)(z-z_0)^{k}$$

$$=\sum_{k=0}^{\infty}a_k(z-z_0)^{k}$$



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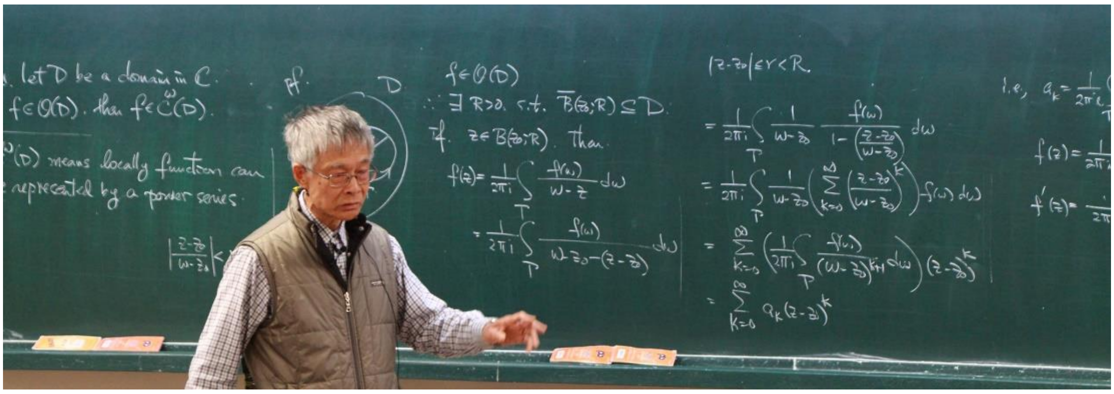

let D be a domain in $\mathbb{C}$.

$f \in \mathcal{O}(D)$. then $f \in \overset{\omega}{C}(D)$.

(D) means locally function can

represented by a power series.

$\left|\frac{z-z_0}{w-z_0}\right| <$

Pf.

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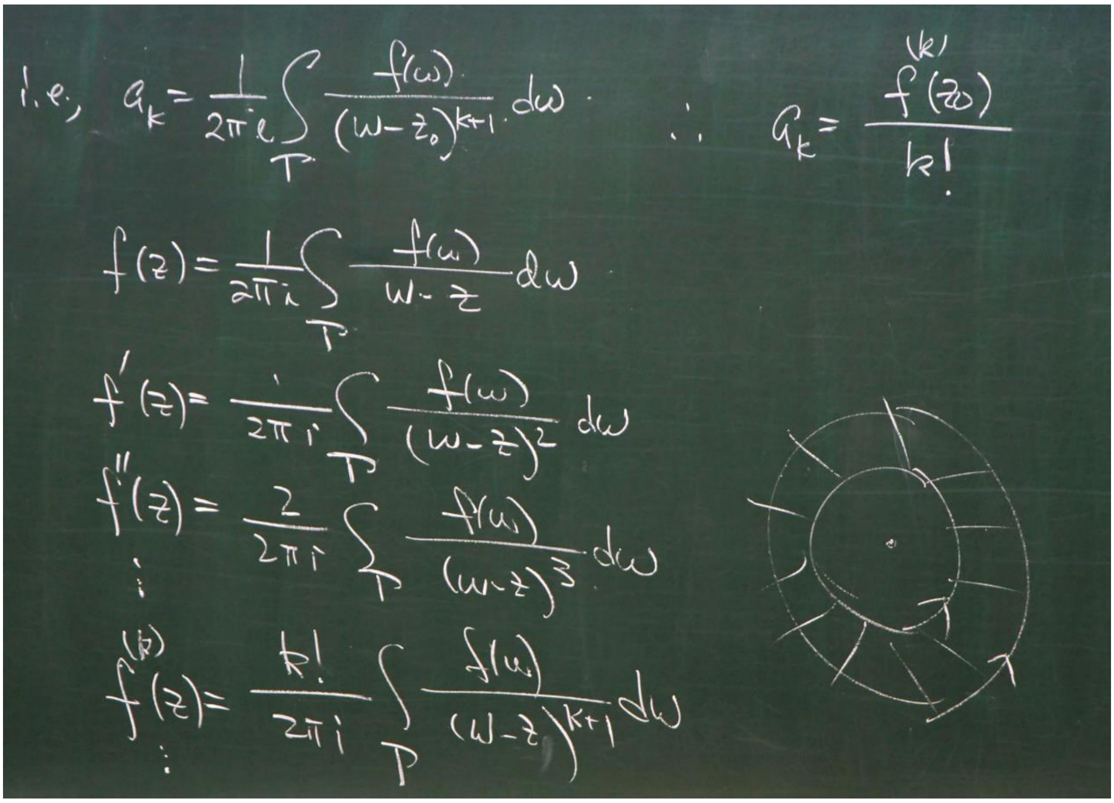
$$= \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

i.e., $a_k = \frac{1}{2\pi i}$

$f(z) = \frac{1}{2\pi}$

$f'(z) =$



i.e., $a_k = \frac{1}{2\pi i}\int_T \frac{f(w)}{(w-z_0)^{k+1}}\,dw$. $\therefore\; a_k = \frac{f^{(k)}(z_0)}{k!}$

$$f(z) = \frac{1}{2\pi i}\int_T \frac{f(w)}{w-z}\,dw$$

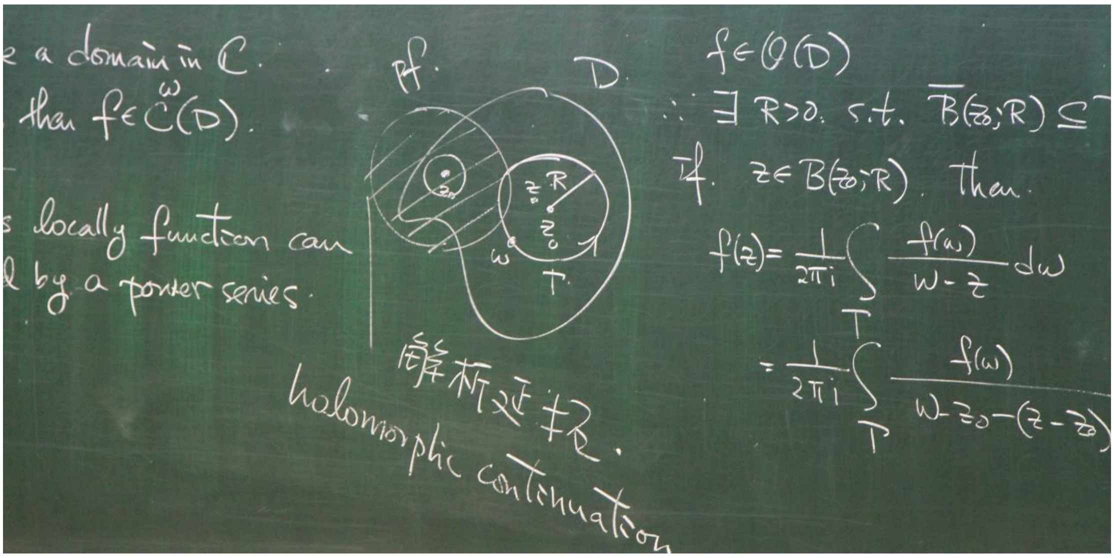
$$f'(z) = \frac{1}{2\pi i}\int_T \frac{f(w)}{(w-z)^2}\,dw$$

$$f''(z) = \frac{2}{2\pi i}\int_T \frac{f(w)}{(w-z)^3}\,dw$$

$\vdots$

$$f^{(k)}(z) = \frac{k!}{2\pi i}\int_T \frac{f(w)}{(w-z)^{k+1}}\,dw$$

$\vdots$



a domain in $\mathbb{C}$.

then $f \in \overset{\omega}{C}(D)$.

locally function can

by a power series.

Pf.

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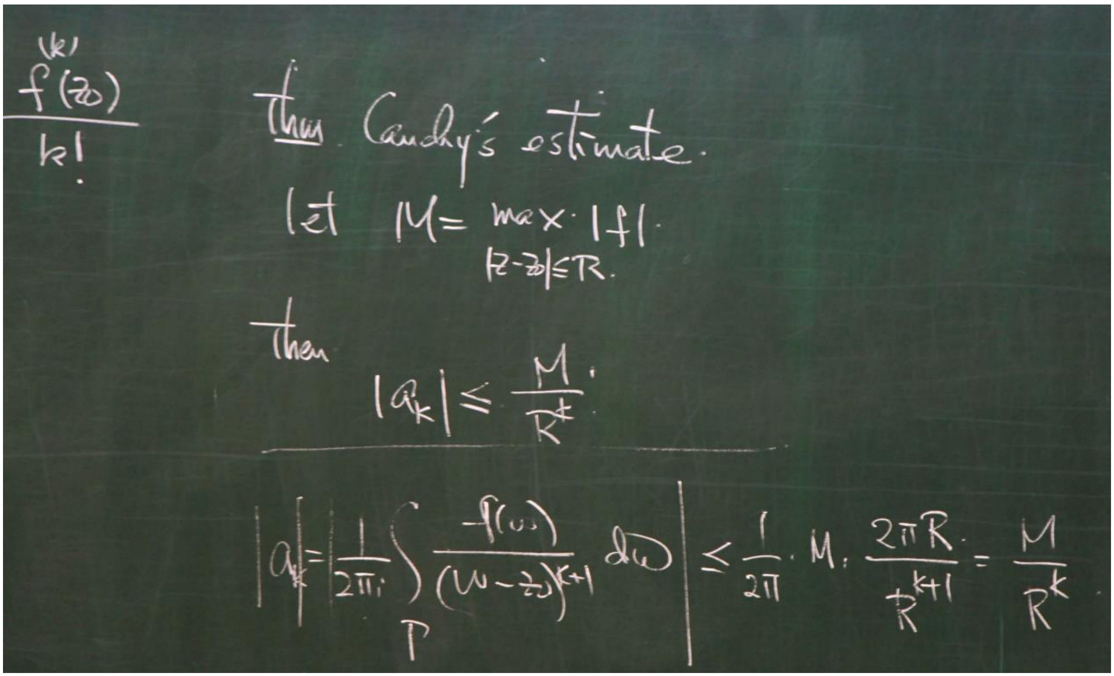
解析延拓.

holomorphic continuation



$$\frac{f^{(k)}(z_0)}{k!}$$

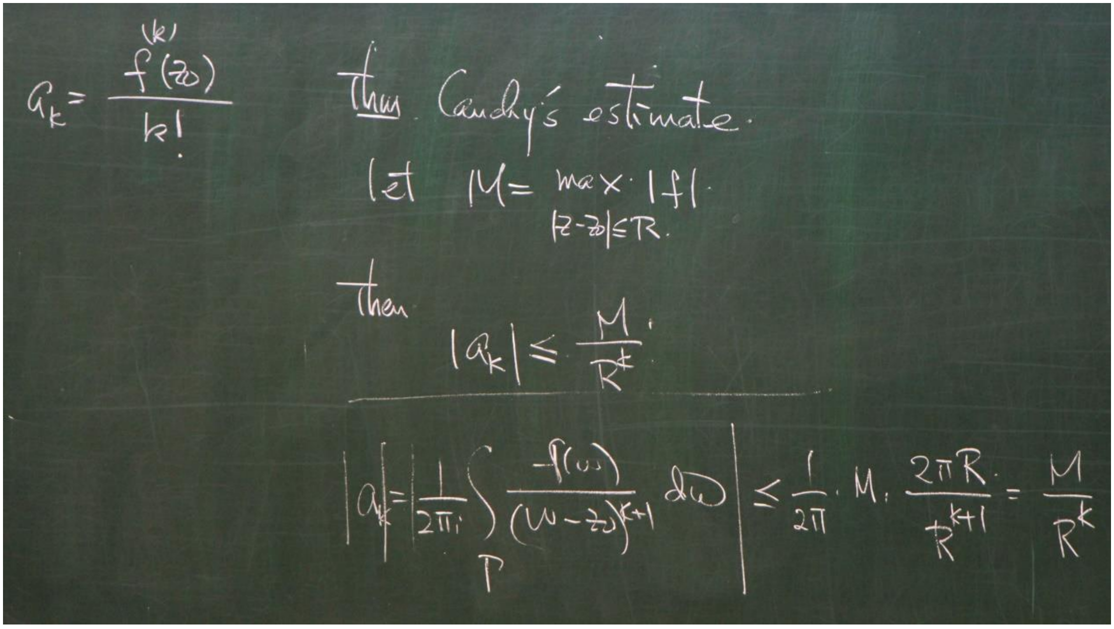
Thm. Cauchy's estimate.

let $M = \max\limits_{|z-z_0|\leq R} |f|$.

Then

$$|a_k| \leq \frac{M}{R^k}.$$

$$|a_k| = \left|\frac{1}{2\pi i}\int_{\Gamma} \frac{f(w)}{(w-z_0)^{k+1}}\,dw\right| \leq \frac{1}{2\pi}\cdot M \cdot \frac{2\pi R}{R^{k+1}} = \frac{M}{R^k}.$$



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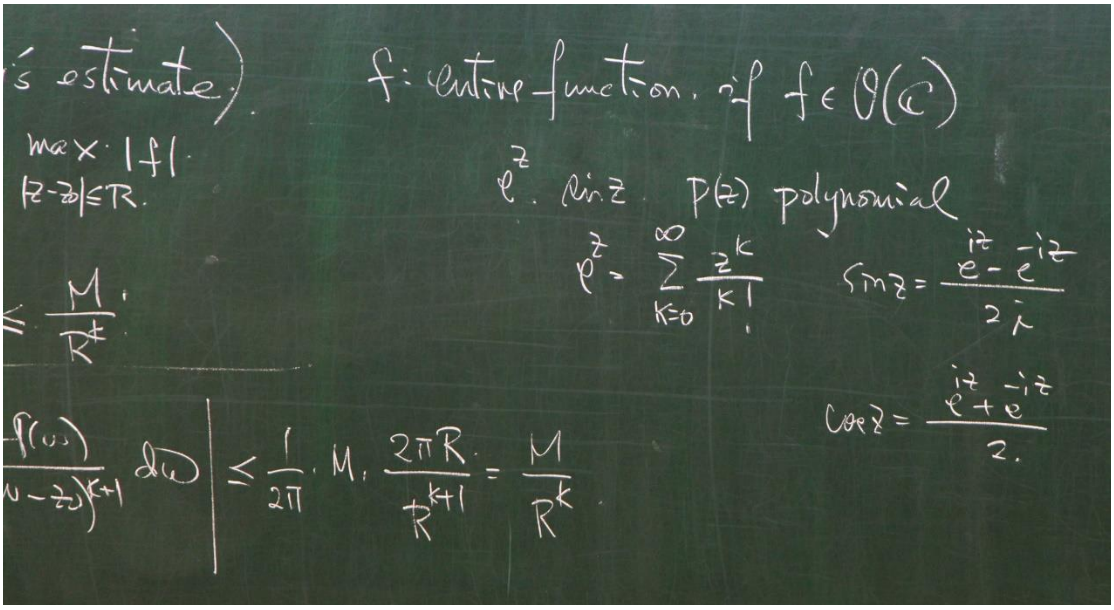

's estimate).

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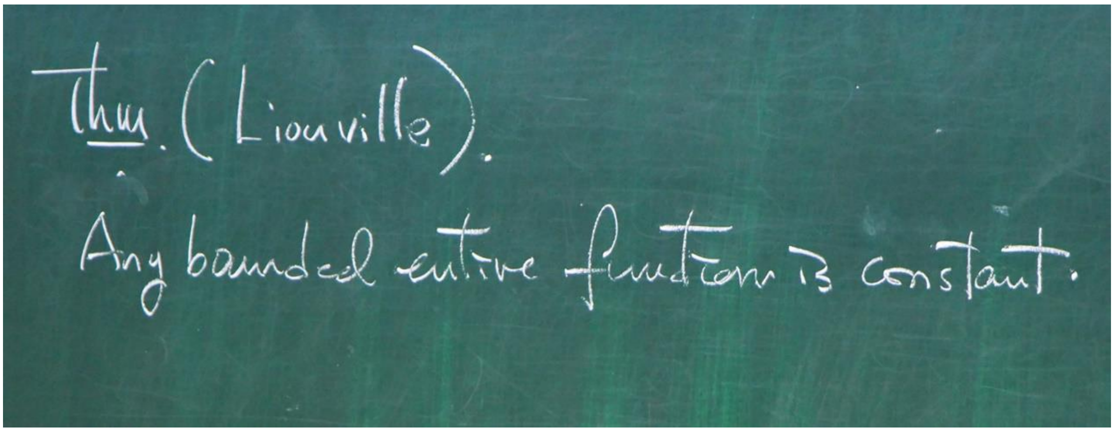
f: entire function, if $f \in \mathcal{O}(\mathbb{C})$

e^z. $\sin z$. $P(z)$ polynomial

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

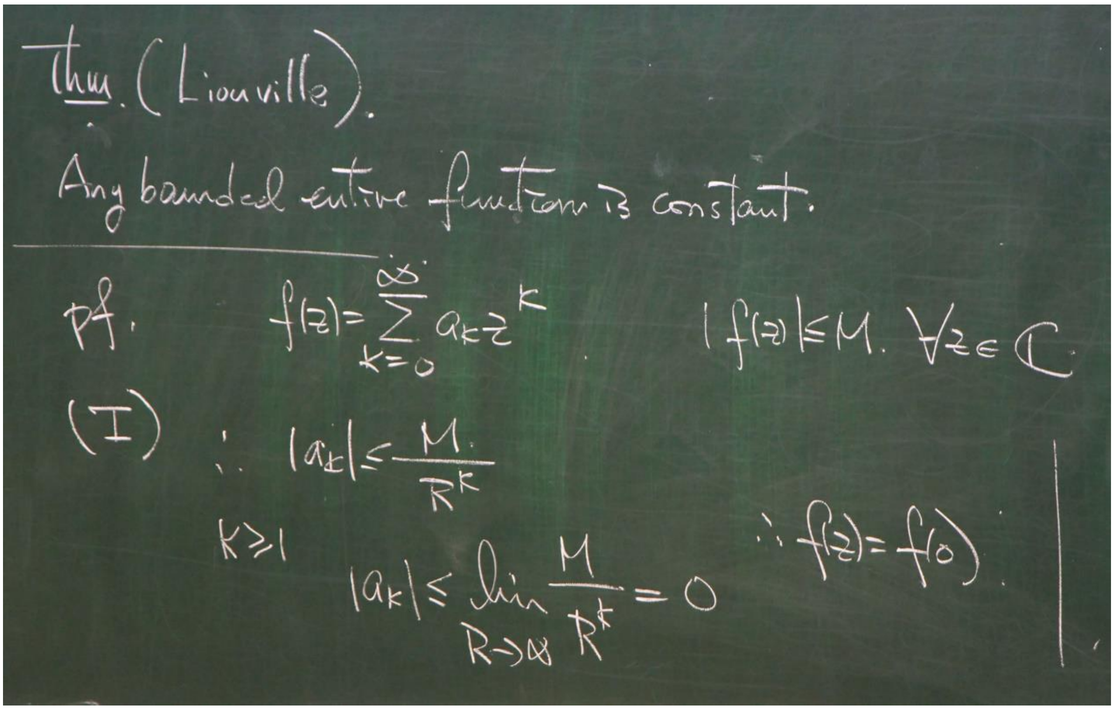
$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}.$$



Thm. (Liouville).

Any bounded entire function is constant.



Thm. (Liouville).

Any bounded entire function is constant.

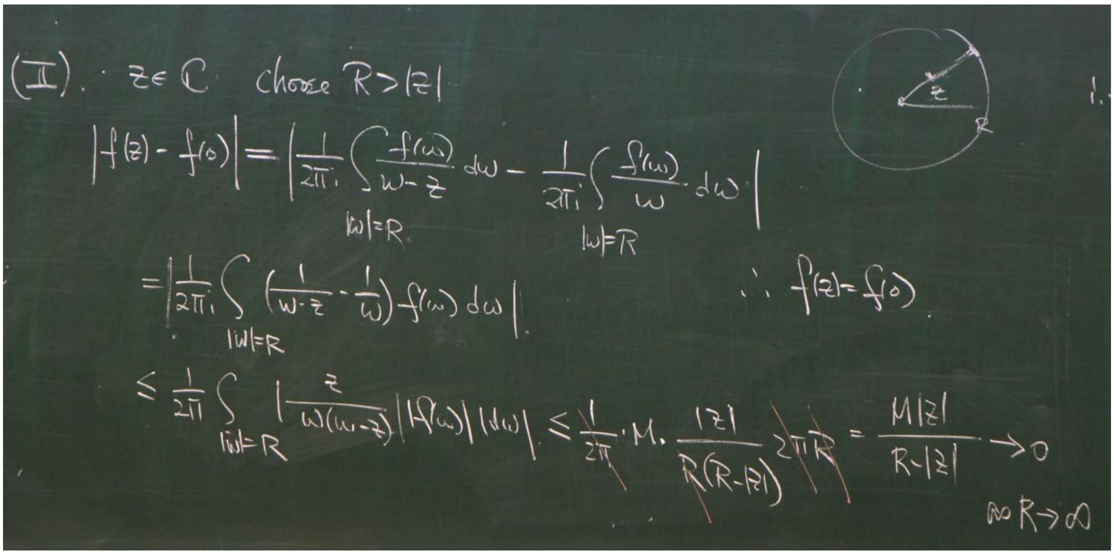
pf. $f(z) = \sum_{k=0}^{\infty} a_k z^k$. $|f(z)| \le M$. $\forall z \in \mathbb{C}$

(I) $\therefore |a_k| \le \frac{M}{R^k}$

$k \ge 1$

$$|a_k| \le \lim_{R\to\infty} \frac{M}{R^k} = 0$$

$\therefore f(z) = f(0)$.



(I). z ∈ ℂ choose $R>|z|$

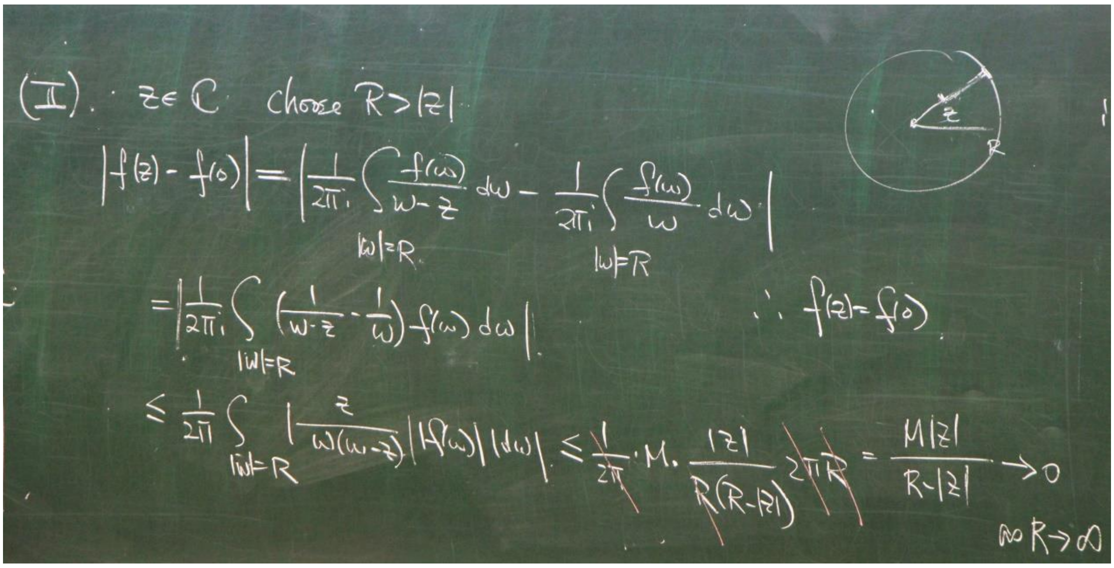
$$|f(z)-f(0)|=\left|\frac{1}{2\pi i}\int_{|w|=R}\frac{f(w)}{w-z}\,dw-\frac{1}{2\pi i}\int_{|w|=R}\frac{f(w)}{w}\,dw\right|$$

$$=\left|\frac{1}{2\pi i}\int_{|w|=R}\left(\frac{1}{w-z}-\frac{1}{w}\right)f(w)\,dw\right|$$

$\therefore f(z)=f(0)$.

$$\leq\frac{1}{2\pi}\int_{|w|=R}\left|\frac{z}{w(w-z)}\right||f(w)|\,|dw|\leq\frac{1}{\cancel{2\pi}}\cdot M\cdot\frac{|z|}{\cancel{R}(R-|z|)}\cancel{2\pi}\cancel{R}=\frac{M|z|}{R-|z|}\to 0$$

as $R\to\infty$



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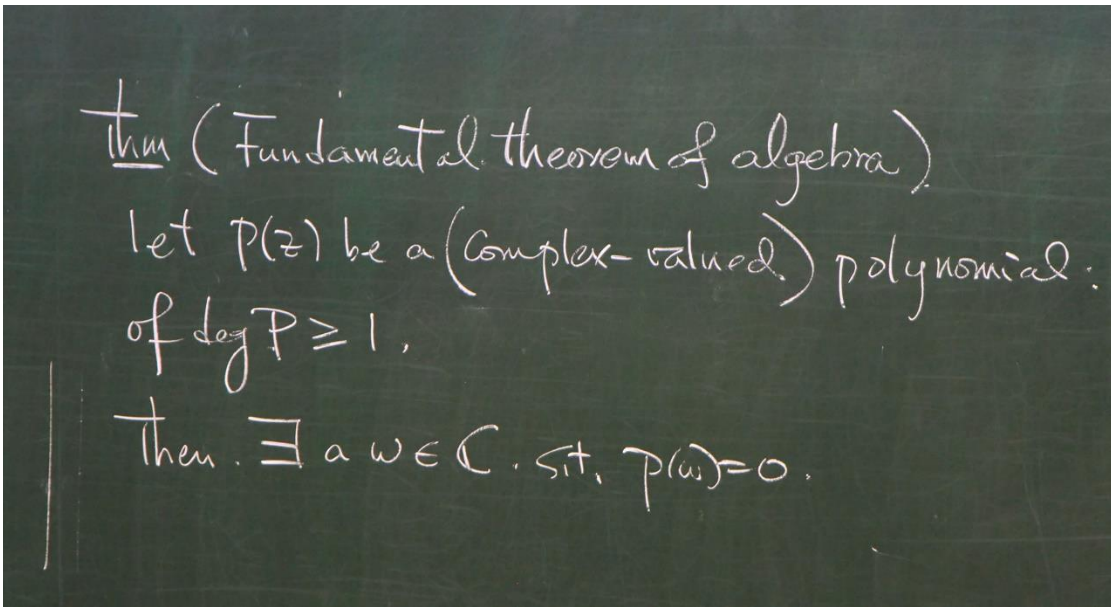
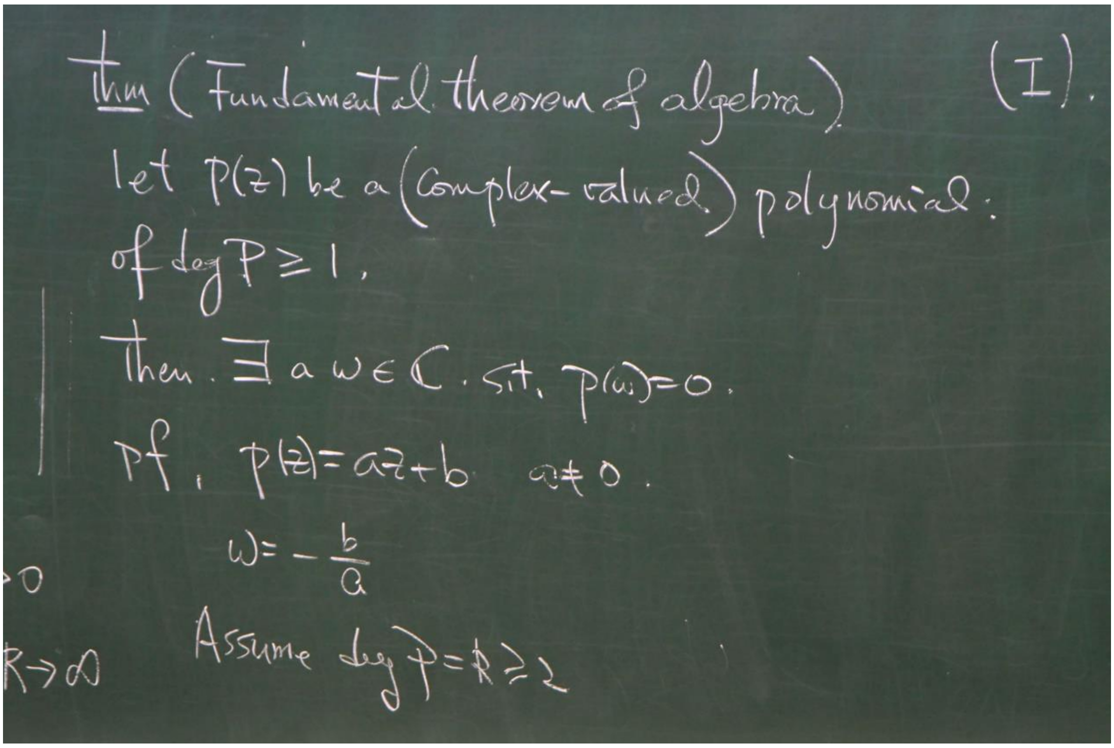


<u>thm</u> (Fundamental theorem of algebra)

let $p(z)$ be a (complex-valued) polynomial of $\deg P \geq 1$,

Then ∃ a $w \in \mathbb{C}$ s.t. $p(w)=0$.



thm (Fundamental theorem of algebra) (I).

let $p(z)$ be a (complex-valued) polynomial: of $\deg P \geq 1$.

Then. $\exists$ a $w \in \mathbb{C}$. s.t. $p(w)=0$.

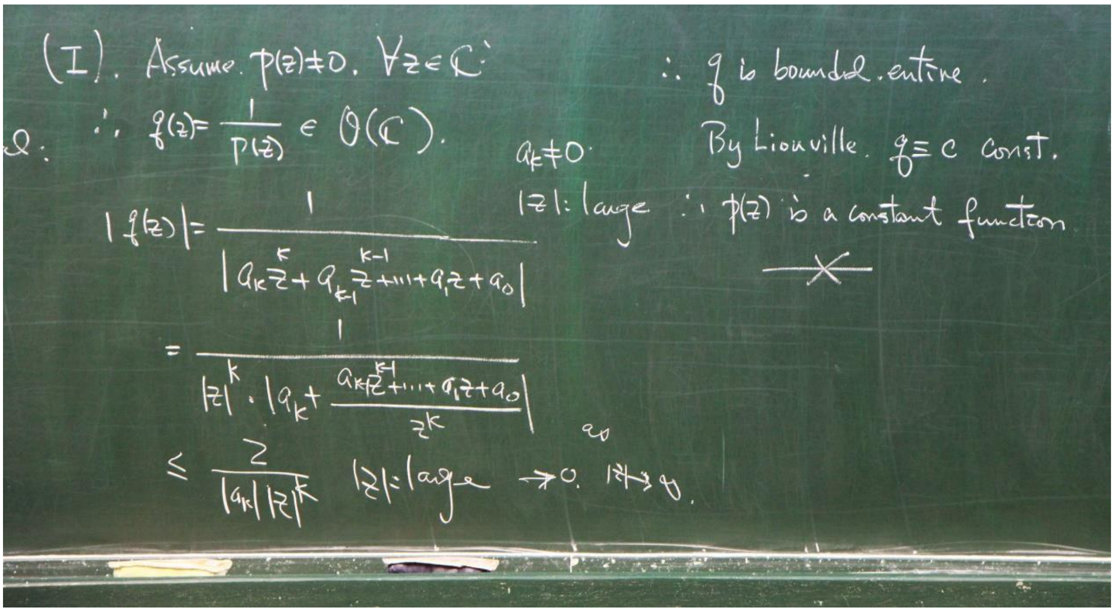
pf. $p(z) = az+b$ $a \neq 0$.

$\to 0$

$$w = -\frac{b}{a}$$

$R \to \infty$

Assume $\deg P = k \geq 2$



(I). Assume. $p(z) \neq 0$. $\forall z \in \mathbb{C}$

$\therefore$ $g(z) = \dfrac{1}{p(z)} \in \mathcal{O}(\mathbb{C})$.

$a_k \neq 0$

$|z|$: lange

$$|g(z)| = \frac{1}{|a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0|}$$

$$= \frac{1}{|z|^k \cdot \left| a_k + \frac{a_{k-1} z^{k-1} + \cdots + a_1 z + a_0}{z^k} \right|}$$

a_0

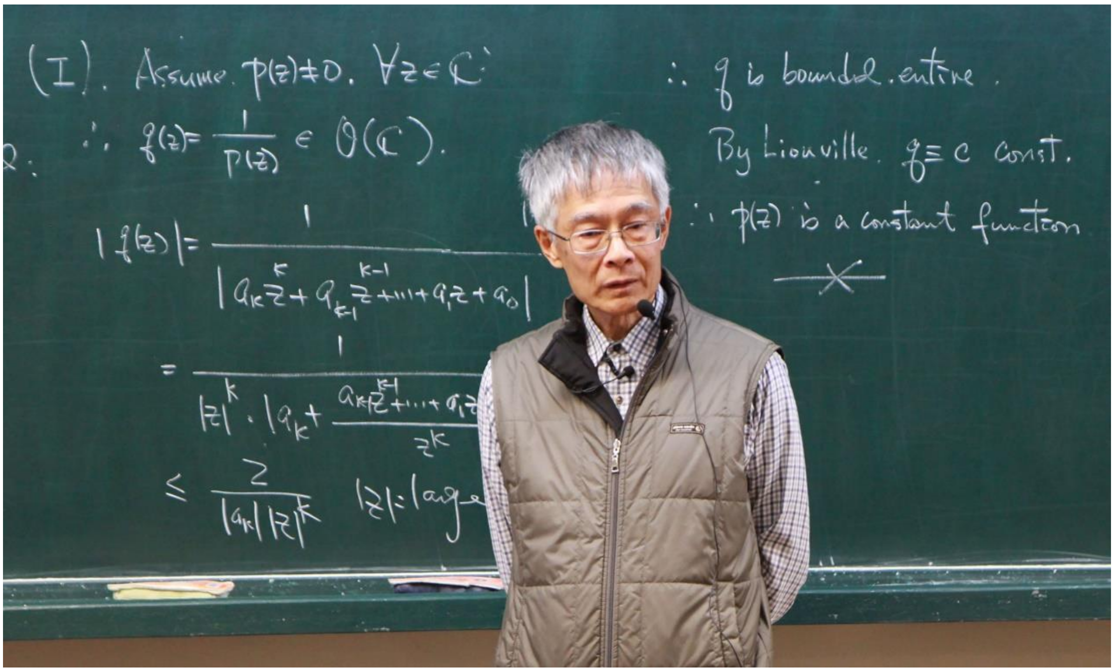
$$\leq \frac{2}{|a_k| |z|^k} \quad |z|: \text{large} \to 0. \ |z| \to \infty.$$

$\therefore$ g is bounded. entire.

By Liouville. $g \equiv c$ const.

$\therefore$ $p(z)$ is a constant function

$\longrightarrow\!\!\!\!\!\times$



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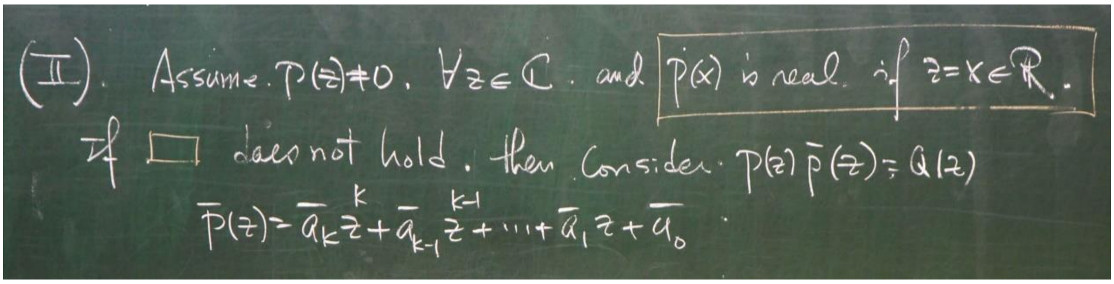
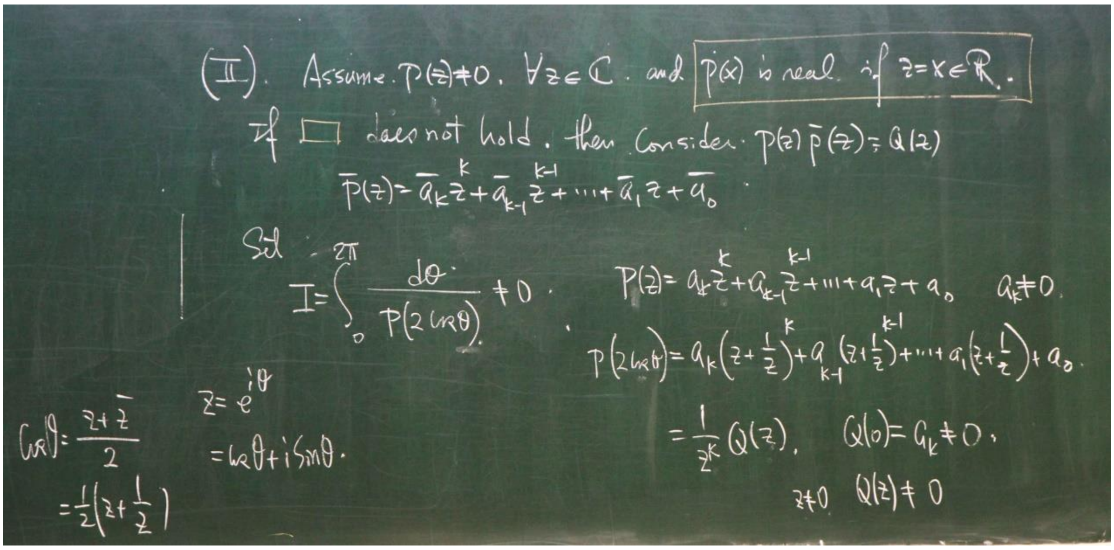


(II). Assume $p(z) \neq 0$. $\forall z \in \mathbb{C}$. and $\boxed{p(x) \text{ is real. if } z = x \in \mathbb{R}.}$

If $\square$ does not hold. then Consider. $p(z)\,\bar{p}(z) = Q(z)$

$$\bar{p}(z) = \bar{a}_k z^k + \bar{a}_{k-1} z^{k-1} + \cdots + \bar{a}_1 z + \bar{a}_0.$$



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Sol

$$I = \int_0^{2\pi} \frac{d\theta}{p(2\cos\theta)} \neq 0.$$

$$p(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0 \quad a_k \neq 0.$$

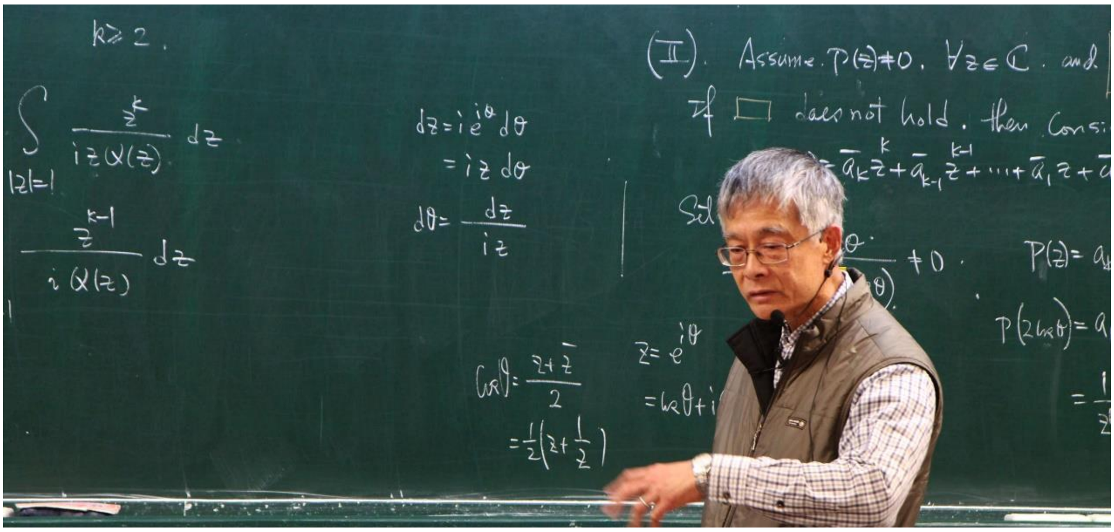
$$p(2\cos\theta) = a_k \left(z + \frac{1}{z}\right)^k + a_{k-1} \left(z + \frac{1}{z}\right)^{k-1} + \cdots + a_1 \left(z + \frac{1}{z}\right) + a_0.$$

$$= \frac{1}{z^k} Q(z), \quad Q(0) = a_k \neq 0.$$

$z \neq 0$ $Q(z) \neq 0$

$$\cos\theta = \frac{z + \bar{z}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$z = e^{i\theta} = \cos\theta + i\sin\theta.$$



k≥2.

$$\int_{|z|=1} \frac{z^k}{iz\,Q(z)}\,dz$$

$$\frac{z^{k-1}}{i\,Q(z)}\,dz$$

$$dz = ie^{i\theta}\,d\theta = iz\,d\theta$$

$$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{z+\bar{z}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$z = e^{i\theta} = \cos\theta + i$$

(II). Assume $p(z)\neq 0$, $\forall z\in\mathbb{C}$. and

If □ does not hold, then consi

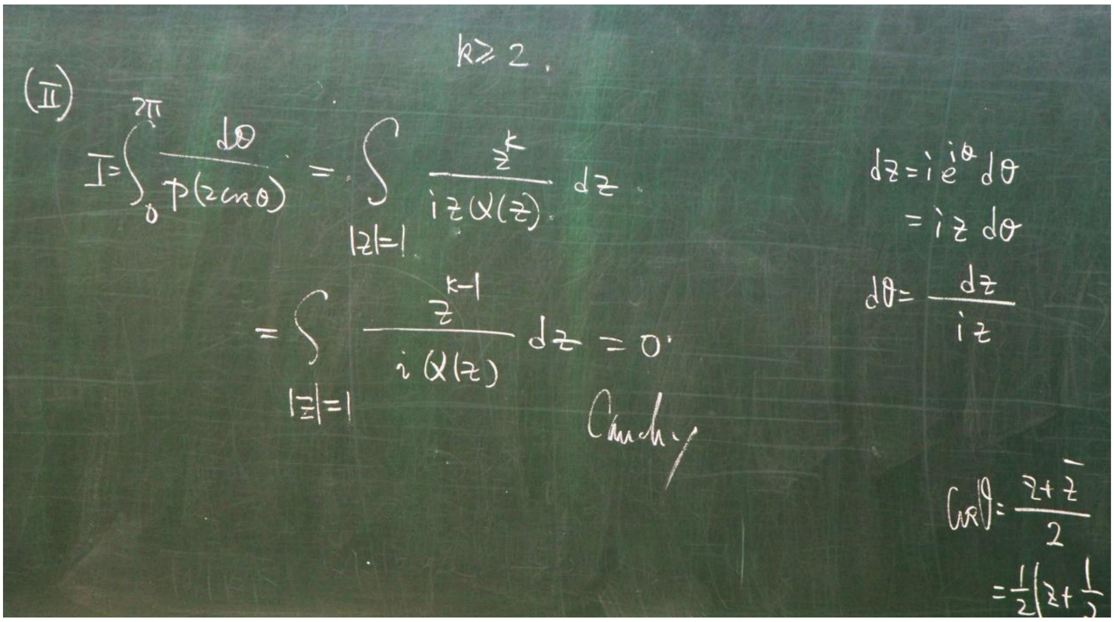
$\bar{a}_k z^k + \bar{a}_{k-1} z^{k-1} + \cdots + \bar{a}_1 z + \bar{a}$

Sol

$\neq 0$.

$P(z) = a$

$p(2\cos\theta) = a$



k≥2.

(II)

$$I = \int_0^{2\pi} \frac{d\theta}{p(2\cos\theta)} = \int_{|z|=1} \frac{z^k}{iz\,Q(z)}\,dz$$

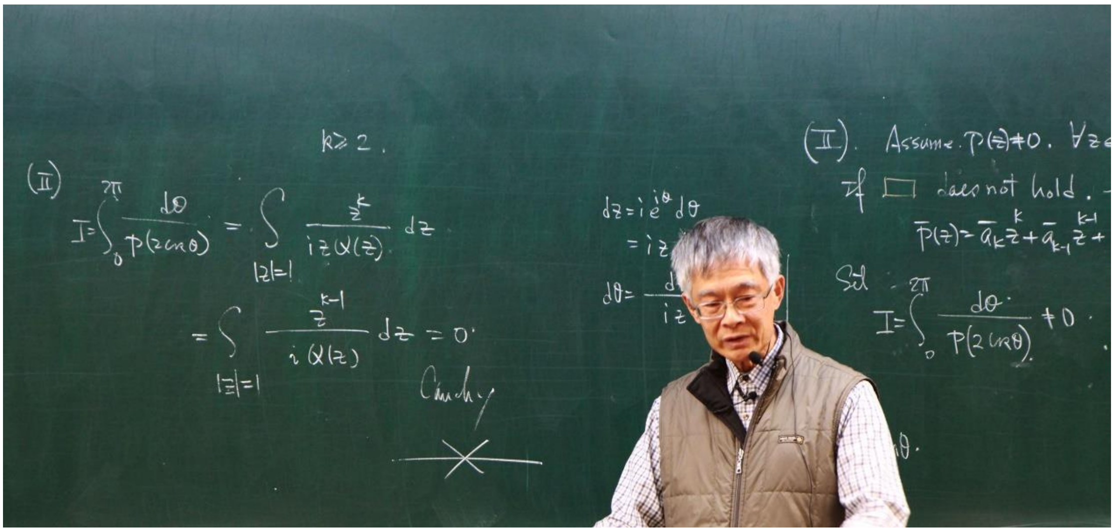
$$= \int_{|z|=1} \frac{z^{k-1}}{i\,Q(z)}\,dz = 0$$

Cauchy

$$dz = ie^{i\theta}\,d\theta = iz\,d\theta$$

$$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{z+\bar{z}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$$



k≥2.

(II)

$$I = \int_0^{2\pi} \frac{d\theta}{p(2\cos\theta)} = \int_{|z|=1} \frac{z^k}{iz\,Q(z)}\,dz$$

$$= \int_{|z|=1} \frac{z^{k-1}}{i\,Q(z)}\,dz = 0$$

Cauchy

$$dz = ie^{i\theta}\,d\theta = iz$$

$$d\theta = \frac{d}{iz}$$

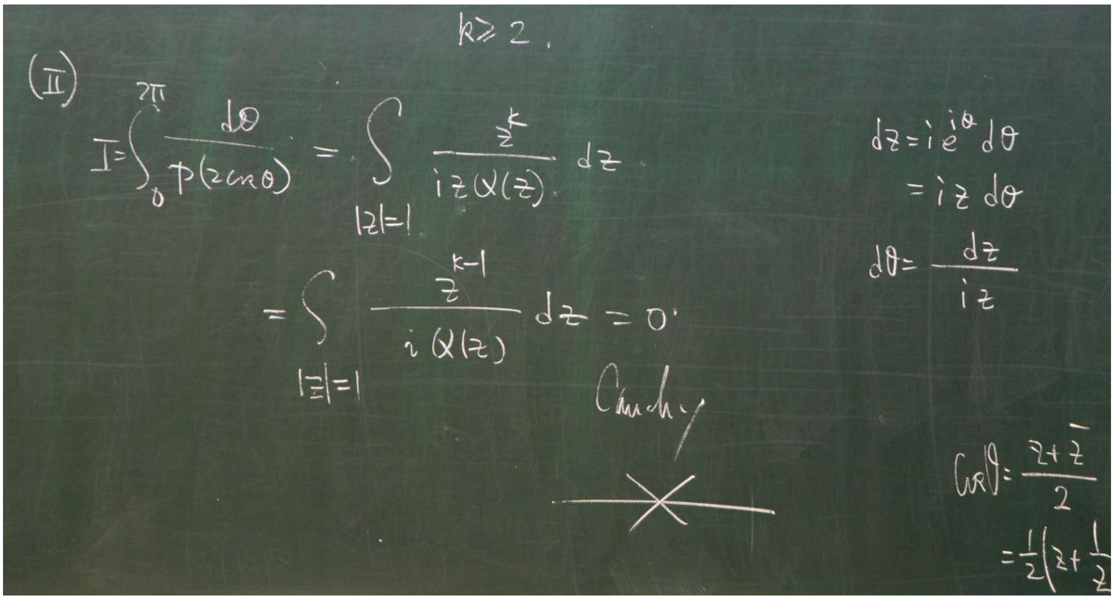
(II). Assume $p(z)\neq 0$, $\forall z$

If □ does not hold.

$\bar{P}(z) = \bar{a}_k z^k + \bar{a}_{k-1} z^{k-1} +$

Sol

$$I = \int_0^{2\pi} \frac{d\theta}{P(2\cos\theta)} \neq 0.$$



$k \geq 2.$

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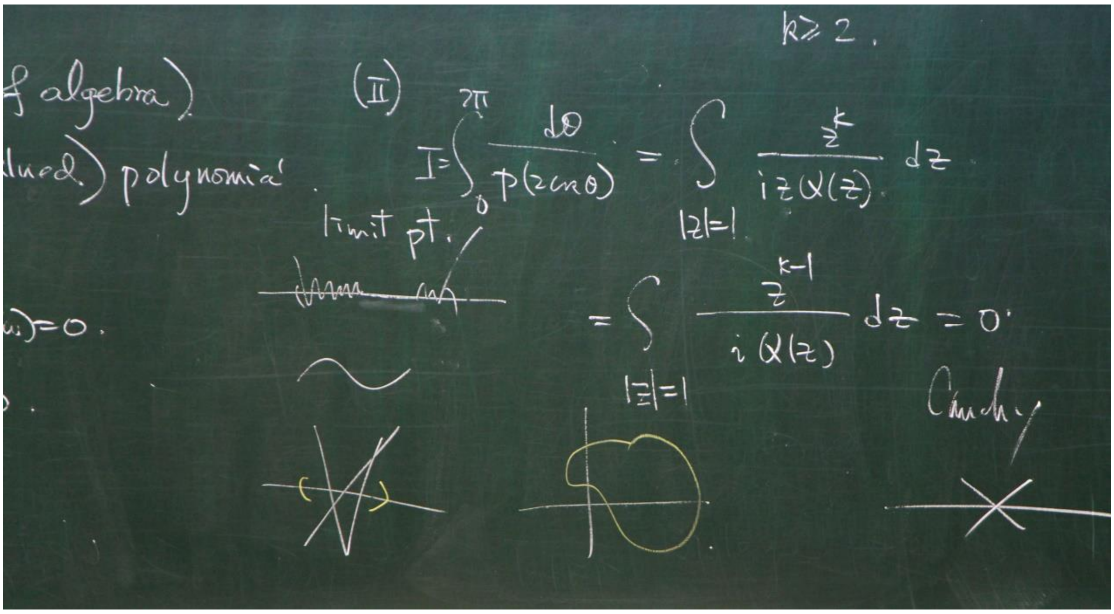

$k \geq 2.$

f algebra)

lued.) polynomial

u)=0.

limit pt.

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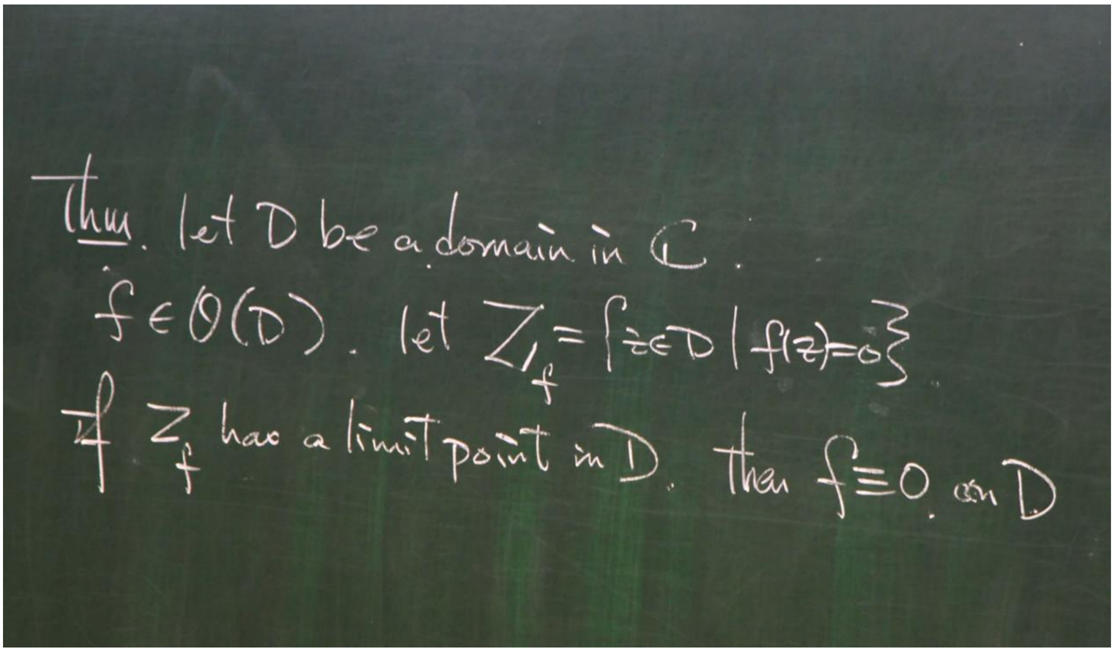
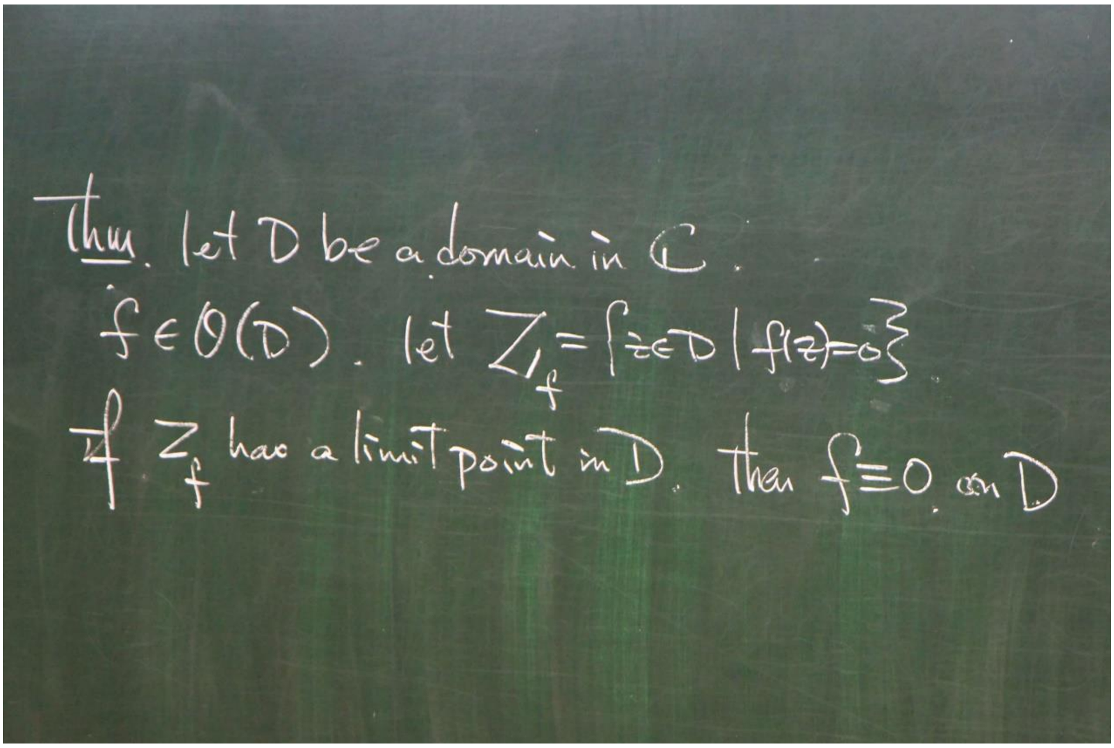


Thm. let D be a domain in $\mathbb{C}$.

$f \in \mathcal{O}(D)$. let $Z_f = \{z \in D \mid f(z)=0\}$

If Z_f has a limit point in D. then $f \equiv 0$ on D



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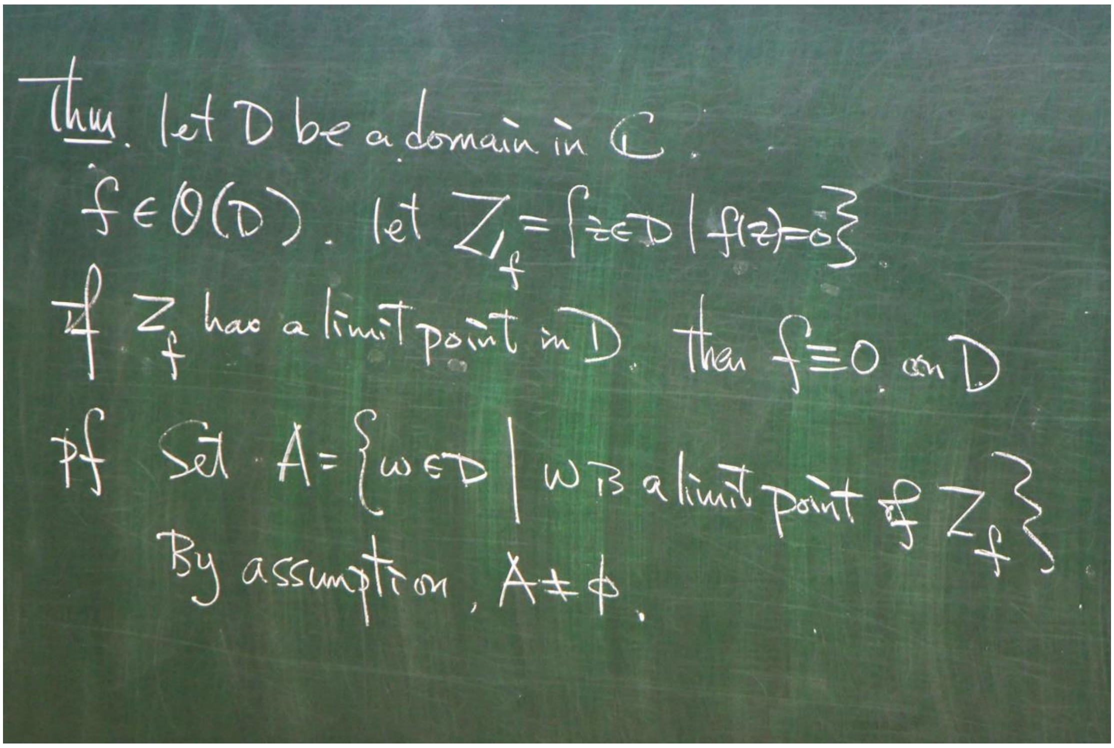

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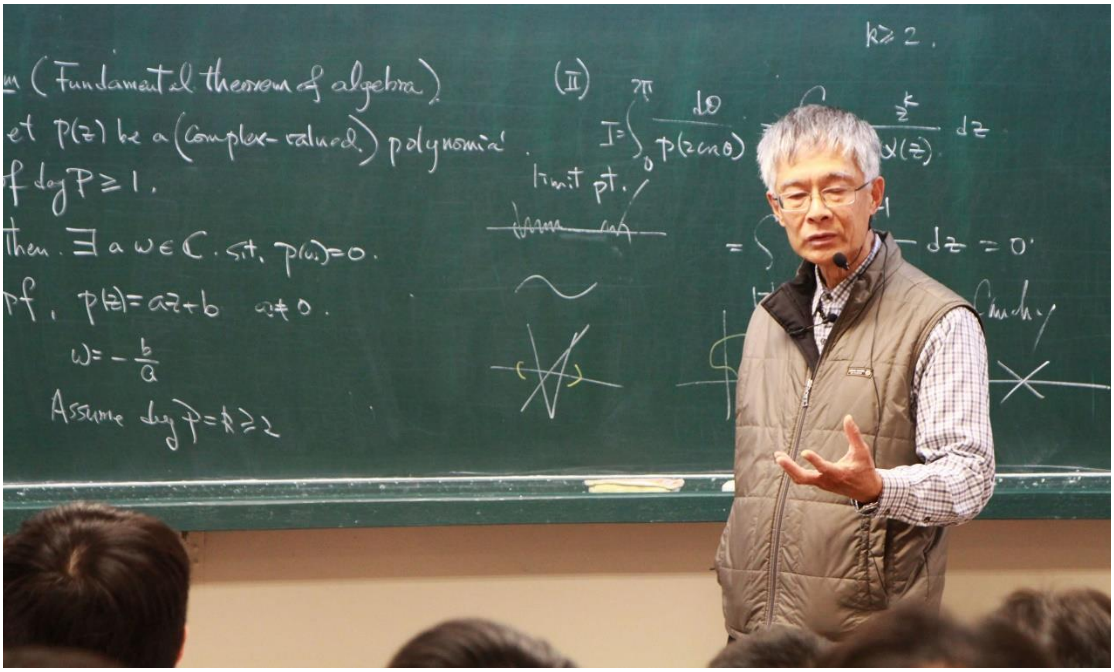
By assumption, $A \neq \phi$.



m (Fundamental theorem of algebra)

et $p(z)$ be a (complex-valued) polynomial.

f $\deg P \geq 1$.

then. $\exists$ a $w \in \mathbb{C}$. s.t. $p(w)=0$.

pf. $p(z)=az+b$ $a \neq 0$.

$w = -\frac{b}{a}$

Assume $\deg P = k \geq 2$

$k \geq 2$.

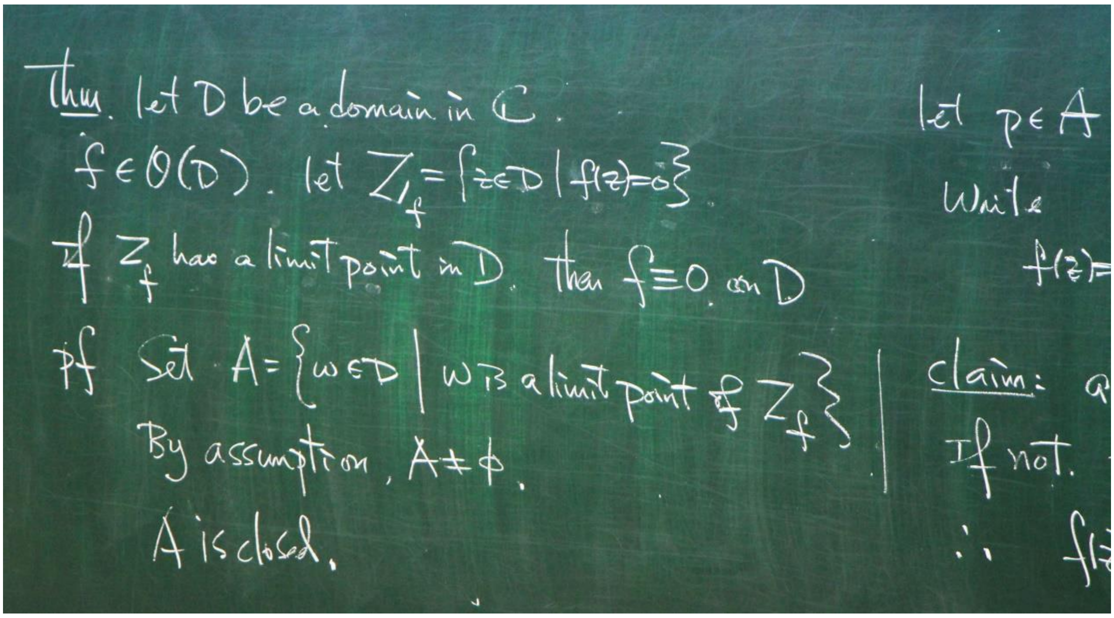
(II)

$I = \int_0^{\pi} \frac{d\theta}{p(2\cos\theta)}$ $\frac{z^k}{q(z)}\,dz$

limit pt.

$= \int$ $dz = 0$

Cauchy



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A is closed.

let $p \in A$

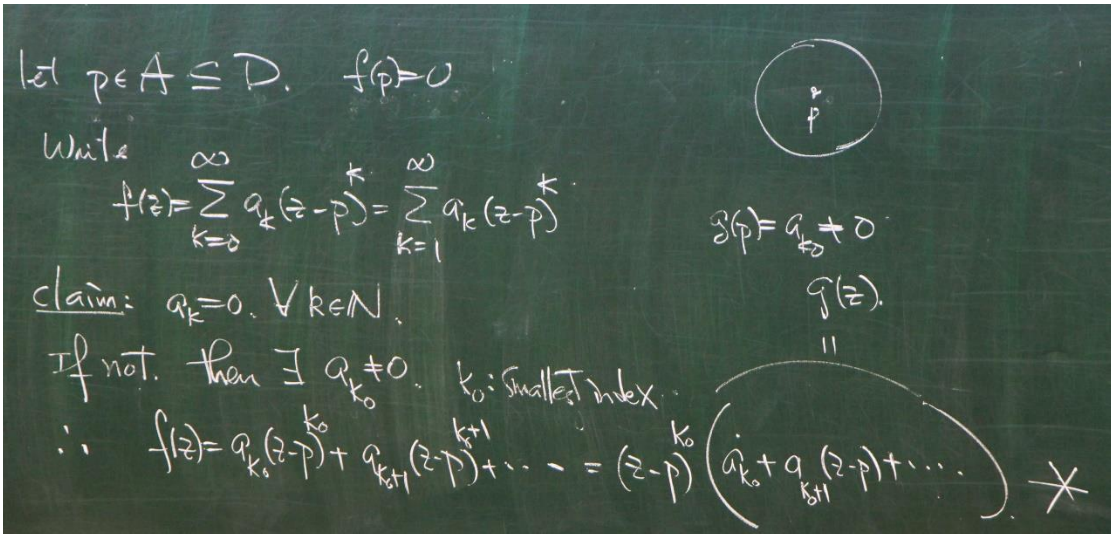
Write

$f(z)=$

claim: a

If not.

$\therefore$ $f(z$



let $p \in A \subseteq D$. $f(p) = 0$

Write

$$f(z) = \sum_{k=0}^{\infty} a_k (z-p)^k = \sum_{k=1}^{\infty} a_k (z-p)^k.$$

claim: $a_k = 0$. $\forall k \in \mathbb{N}$.

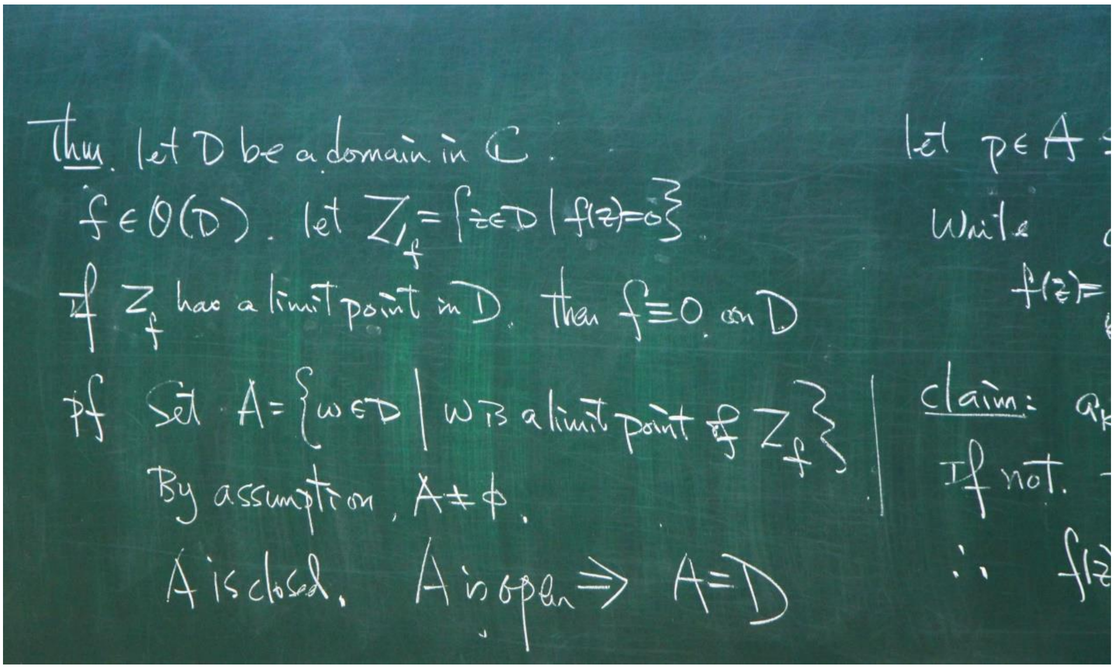
If not. then $\exists\ a_{k_0} \neq 0$. k_0: Smallest index.

$$\therefore \quad f(z) = a_{k_0}(z-p)^{k_0} + a_{k_0+1}(z-p)^{k_0+1} + \cdots = (z-p)^{k_0} \underbrace{\left(a_{k_0} + a_{k_0+1}(z-p) + \cdots\right)}_{}. \ \times$$

$g(p) = a_{k_0} \neq 0$

$g(z)$.

$\shortparallel$



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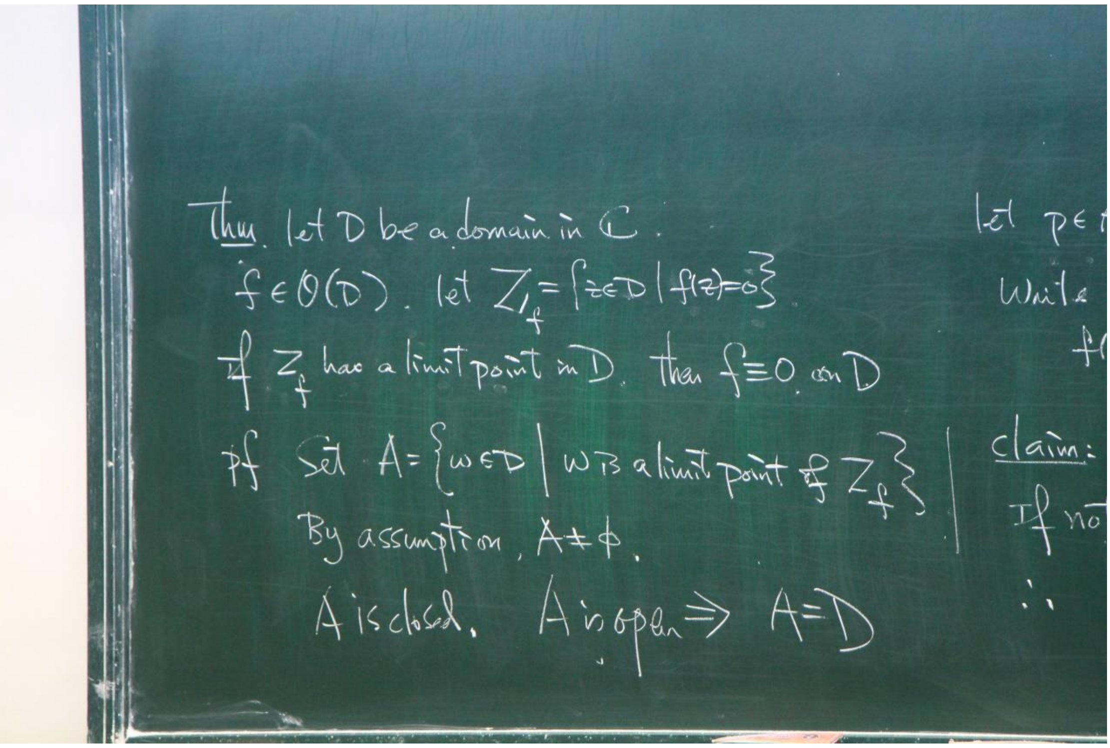

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